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Technical Notes

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Solution to Heat Equation Inside Cryogenic Vessels Using Boubaker Polynomials

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Nomenclature

c	=	specific heat capacity, $J \cdot Kg^{-1} \cdot K^{-1}$
dS	=	heat transfer surface area, m^2
h	=	convection heat transfer coefficient, $W \cdot m^{-2} \cdot K^{-1}$
k	=	thermal conductivity, $W \cdot m^{-1} \cdot K^{-1}$
M	=	fluid molecular weight, N
N	=	prefixed integer
p	=	fluid pressure at mean temperature, Pa
Q_{conv}	=	exchanged convective heat, J
R	=	gas universal constant
T	=	absolute temperature, K
$\bar{T}$	=	dimensionless temperature
T_b	=	hydrogen boiling point temperature, K
T_m	=	fluid mean absolute temperature, K
T_∞	=	room absolute temperature, K
T_0	=	fluid initial absolute temperature, K
α	=	fluid–steel transfer coefficient
α_q	=	Boubaker polynomials' minimal positive roots (dimensionless)
γ	=	heat capacity ratio (dimensionless)
ξ_q	=	real coefficients (dimensionless)
ρ	=	density,$Kg \cdot m^{-3}$
ϖ	=	convective transfer velocity, $m\,s^{-1}$

I. Introduction

RECENTLY, the use of hydrogen as an energy vector has tremendously increased [1–5]. Its abundance and pollution-free combustion caused research on its production, stocking, and conveying to thrive spectacularly in a short period. With this movement, the stocking problem became more and more important. In fact, despite its low boiling point, commercial and transport constraints compelled the handling of hydrogen in a liquid phase ($T_b \approx 20$ K).

The storage problem is, hence, a challenging issue that cuts across applications and the delivery of hydrogen as a renewable energy carrier. The cryogenic vessels (or cryostats) used as a solution to store and transport liquid hydrogen are generally metallic double-walled vessels with an inserted high-vacuum insulation.

In this technical note, we propose an analytical solution to the convective heat equation inside a standard cylindrical vessel. The main problem formulation is based on the similarities between the convective heat equation and the Boubaker polynomials' characteristic differential equation.

II. Vacuum-Insulated Hydrogen Cylindrical Vessel Model

It is known that, in addition to pipelines, hydrogen can be transported inside bridging compounds such as ammonia, metal hydrides, or methanol. Nevertheless, the most adopted method is transport under cryogenic conditions.

The following two transportations vessels types are presented in Fig. 1: 1) a metallic cylinder (Fig. 1a) containing liquid hydrogen kept at a temperature of $T_0 \approx 10$ K, and 2) a vacuum cylindrical vessel (Fig. 1b).

The room temperature for both mountings is supposed to be constant ($T_\infty \approx 290$ K).

It was demonstrated in the first case (Fig. 1a) that the convective flux toward the liquid is predominant. The amount of heat Q_{conv} can be estimated using Newton's cooling law:

$$Q_{\mathrm{conv}} = h\Delta T = h(T_\infty - T_0) \tag{1}$$

For a standard steel alloy compartment ($h = 6.5\ W \cdot m^{-2} \cdot K^{-1}$), the extremely high amount of heat ($Q_{\mathrm{conv}} \approx 2450\ W \cdot m^{-2}$) causes the liquid hydrogen to boil after a y short time ($t'_b \approx 2.1 \times 10^3\ s \approx 36$ min).

The vacuum-insulated cylindrical model, as shown in Fig. 1b, consists of cylindrically shaped double-wall vessels. The shell materials are stainless steel or similar alloys prepared for supporting a high vacuum with a pressure of less than 0.001 torr.

In this model, both conduction and convection are to be taken into account because the radiation effect is meaningless at such low temperatures, as mentioned by Gause and McKannan [6].

The convective heat amount, Q_{conv}, can be calculated as follows:

$$Q_{\mathrm{conv}} = a\frac{\gamma+1}{\gamma-1} \cdot \sqrt{\frac{R}{8\pi \cdot T_m \cdot M}}p \cdot dS \cdot (T - T_\infty) \tag{2}$$

By introducing the dimensionless variable $\bar{T}$, $\bar{T} = (T - T_\infty)/(T_0 - T_\infty)$, we have

$$\begin{cases} Q_{\mathrm{conv}} = \rho c \varpi \bar{T} \\ \varpi = \frac{a(T_0 - T_\infty)}{\rho c}\frac{\gamma+1}{\gamma-1} \cdot \sqrt{\frac{R}{8\pi \cdot T_m \cdot M}}p \cdot dS \end{cases} \tag{3}$$

The heat equation in the vacuum area is, hence,

$$c\frac{\partial(\rho\bar{T})}{\partial t} + \nabla \cdot (-k\nabla\bar{T} + \rho c \varpi \bar{T}) = 0 \tag{4}$$

Equation (4) alters to

$$\frac{\partial \bar{T}}{\partial t} = D\frac{\partial^2 \bar{T}}{\partial x^2} - \varpi\frac{\partial \bar{T}}{\partial x} \tag{5}$$

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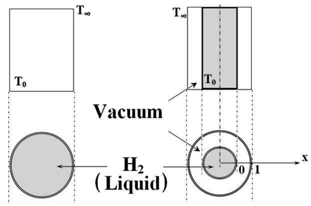

Fig. 1 Transportation vessels types.

III. Boubaker Polynomials

The earliest monomial definition of the Boubaker polynomials [7–10] appeared in a physical study that yielded an analytical solution to the heat equation inside a physical model [10]. This monomial definition is expressed as follows:

$$\hat{B}_n(X) = \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] \cdot (-1)^p \cdot X^{n-2p} \tag{6}$$

where

$$\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4} \tag{7}$$

where brackets designate the floor function.

The Boubaker polynomials have the ordinary generating function:

$$f_{\hat{B}}(X,t) = \frac{1+3t^2}{1+t(t-2X)} \tag{8}$$

They are also defined by the following recursive system:

$$\begin{cases} \hat{B}_4(X) = X^4 - 2; \\ \hat{B}_8(X) = X^8 - 4X^6 + 8X^2 - 2; \\ \hat{B}_{4(q+1)} = (X^4 - 4X^2 + 2) \cdot \hat{B}_{4(q)} - \hat{B}_{4(q-1)} \end{cases} \tag{9}$$

Since 2004, the Boubaker polynomials' expansions have been successfully used in several applied physics studies, that is, the models presented by Awojoyogbe and Boubaker in the field of organic tissues modelling [7] and the works of Ghanouchi et al. on heat transfer modeling systems [8].

IV. Convective Heat Equation Solution

To solve Eq. (5), a dimensionless separable variables function, $\bar{T}(x,t)$, is defined as follows:

$$\bar{T}(x,t) = \frac{T(x,t) - T_\infty}{T_0 - T_\infty} = T_1(t) \cdot T_2(x) \tag{10}$$

where T_1 is a t-dependent term, and T_2 is the spatial component expressed as a polynomial expansion:

$$T_2(x) = \frac{1}{2N} \sum_{q=1}^{N} \xi_q \cdot \hat{B}_{4q}(x \cdot \alpha_q) \tag{11}$$

where α_q are the minimal positive roots of the Boubaker polynomials [10], $\hat{B}_{4q}$.

The final solution is a finite sum of elementary solutions:

$$\bar{T}(x,t) = \sum_{q=1}^{N} \bar{T}_q(x,t) = \frac{1}{2N} \sum_{q=1}^{N} \xi_q \cdot T_1(t) \cdot \tilde{B}_{4q}(x \cdot \alpha_q) \tag{12}$$

Parallel to these definitions, Eq. (9) is altered to

$$\hat{B}''_{4q}(X) - \frac{3X(4qX^2 + 12q - 2)}{(1-X^2)(12qX^2 + 4q - 2)} \hat{B}'_{4q}(X) - 4q \frac{(3X^2n^2 + n^2 - 6n + 8)}{(1-X^2)(nX^2 + 3n - 2)} \hat{B}_n(X) = 0 \tag{13}$$

then to

$$\begin{cases} \hat{B}''_{4q}(X) - h_q(X) \cdot \hat{B}'_{4q}(X) = g_q(X) \cdot \hat{B}_{4q}(X) \\ h_q(X) = \frac{3X(4qX^2+12q-2)}{(1-X^2)(12qX^2+4q-2)} \\ g_q(X) = 4q \frac{(48X^2q^2+16q^2-24q+8)}{(1-X^2)(12qX^2+4q-2)} \end{cases} \tag{14}$$

The main heat transfer equation becomes

$$T'_1(t) \cdot T_2(x) = D\Big(T_1(t) \cdot T''_2(x) - (u/D) T_1(t) \cdot T'_2(x) \Big) \tag{15}$$

and

$$T'_1(t) \cdot T_2(x) = T_1(t) \cdot D \Big(T''_2(x) - (u/D) T'_2(x) \Big) \tag{16}$$

Finally, by identifying the terms between Eqs. (14) and (16), we obtain

$$T'_1(t) = T_1(t) \cdot D \cdot \left(4q \frac{(48x^2q^2 + 16q^2 - 24q + 8)}{(1-x^2)(12qx^2 + 4q - 2)} \right) \tag{17}$$

which gives the following solution:

$$\bar{T}(x,t) = \sum_{q=1}^{N} \bar{T}_q(x,t) = \frac{1}{2N} \sum_{q=1}^{N} \xi_q e^{-D \cdot 4q \frac{(24x^2q^2+8q^2-12q+4)}{(1-x^2)(6qx^2+2q-1)} t} \cdot \tilde{B}_{4q}(x \cdot \alpha_q) \tag{18}$$

The relevant initial conditions are

$$\bar{T}(x,t)|_{t=0,x=0} = 1 \tag{19}$$

$$\left. \frac{\partial \bar{T}(x,t)}{\partial x} \right|_{t=0,x=0} = 0 \tag{20}$$

$$\bar{T}(x,t)|_{t=0,x=1} = 0 \tag{21}$$

$$\left. \frac{\partial \bar{T}(x,t)}{\partial x} \right|_{t=0,x=1} \neq 0 \tag{22}$$

The conditions expressed by Eqs. (20–22) are already satisfied by the proposed expansion [Eq. (12)]. The remaining condition (19) induces the mathematical constraint

$$-\frac{1}{N} \sum_{q=1}^{N} \xi_q = 1$$

where the condition is satisfied, that is, for $\xi_1 = \xi_2 = \xi_3 = \cdots = \xi_N = -1.0$.

The graphics of the solution to Eqs. (18–22) are given in Fig. 2.

V. Conclusions

The obtained initial profile (at $t = 0$) conforms to the experimental conditions. The following dependent evolution can be compared with other dynamical profiles [11,12].

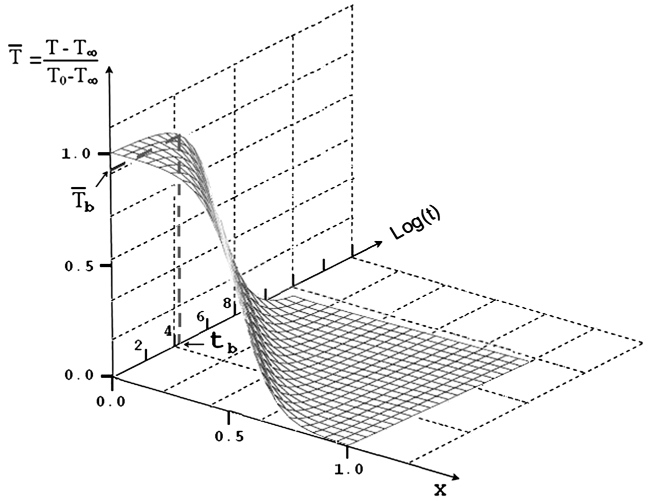

Fig. 2 Three-dimensional representation of the solution $\bar{T}(x, t)$.

Another existing feature is also verified: the use of the cryogenic vessels increases the boiling time 10^2 times. In fact, we noticed that the boiling time, t_b, is about 1.58×10^4 s.

The proposed canonical expansion presents a methodology for works aimed at solving applied physics equations. In fact, by means of the Boubaker polynomials' arithmetical proprieties, the Cauchy boundary conditions are intrinsically verified and indirectly taken into account inside the main equation. The canonical expansion presented in this note is being tested as part of several actual applied physic models.

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